$\mathbf{Z}_2$ as the Topological Origin of
B+L Violation in the Hot Electroweak Theory

Minos Axenides[1]

University of Crete, Department of Physics
GR-71409 Iraklion-Crete, Greece

Andrei Johansen[2]

Harvard University, Lyman Laboratory of Physics
Cambridge, MA 02138, USA

Holger Bech Nielsen[3]

The Niels Bohr Institute, University of Copenhagen
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Ola Törnkvist[4]

NORDITA
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Abstract

The space of static finite-energy configurations in the electroweak theory admits a $\mathbf{Z}_2$ topological structure. More precisely, we show that this space contains two disconnected sectors of unstable gauge-Higgs fields odd under a properly defined generalized parity. This classification extends the description of baryon and lepton number violating electroweak processes to the symmetric phase of the theory. Configurations with odd pure-gauge behaviour at spatial infinity, such as the sphaleron, multisphalerons, electroweak strings as well as an infinite surface of their equivalents, have half-integer Chern-Simons number and mediate $B + L$ violating processes in the early universe. Finite-energy configurations with even pure-gauge behaviour, such as the S^* new sphaleron and electroweak strings, are topologically equivalent to the vacuum and are irrelevant for $B + L$ violation. We discuss the possible formation of $B + L$ violating quark-lepton condensates in the symmetric high-temperature phase of the electroweak theory.

[1]e-mail: `axenides@nbivax.nbi.dk`

[2]e-mail: `johansen@string.harvard.edu`, `ajohansen@nbivax.nbi.dk`

[3]e-mail: `hbech@nbivax.nbi.dk`

[4]Present address: NASA/Fermilab Astrophysics Center, MS-209, P.O. Box 500, Batavia, IL 60510, USA; e-mail: `olat@fnas13.fnal.gov`

1 Introduction

Electroweak interactions are well described by models with chiral fermions and an $SU(2) \times U(1)$ gauge symmetry broken spontaneously to $U(1)$. The interaction violates the conservation of baryon (B) and lepton (L) quantum numbers through the chiral anomaly. As a consequence, the possibility of generating the baryon-number asymmetry of the universe at the electroweak phase transition has recently received increased attention [1].

At zero temperature $B+L$ violation is negligible and is a consequence of quantum tunneling between the distinct vacua which constitute the topologically nontrivial periodic structure of the $SU(N)$ Yang-Mills vacuum. The tunneling is induced by instantons in whose background the four-dimensional Dirac operator has normalizable zero modes [2]. The nontrivial structure is a result of the existence of a homotopy group $\pi_3(SU(N)) = \mathbf{Z}$.

At non-zero temperature the density of instantons is believed to be suppressed due to biscreening [3]. Yet it has been argued [4] that for temperatures below the $SU(2) \times U(1)$ phase transition temperature T_{crit}, the thermal $B + L$ violating rates are dominated by "sphaleron" saddle point solutions [5, 6] to the electroweak field equations. These solutions are static unstable finite-energy gauge-Higgs field configurations with a Chern-Simons number $CS = 1/2$ (modulo 1) relative to the vacuum. Their existence is a consequence of the periodic vacuum structure. More precisely, a sphaleron configuration is the point of highest energy ($E_{sph} \approx M_W/\alpha$) on the lowest-energy continuous path of static field configurations that leads across the potential energy barrier and interpolates between the topologically distinct vacua with $CS = n$ and $CS = n + 1$ ($n \in \mathbf{Z}$). $B + L$ violating transitions across the sphaleron saddle point are due to the crossing of fermion energy levels, which is a consequence of the existence of normalizable zero-energy solutions to the three-dimensional Dirac equation in the sphaleron background [7].

The existence of the electroweak sphaleron is also a manifestation of the nontrivial structure of the infinite-dimensional space of static finite-energy configurations of the electroweak theory. Such a structure is hinted to by the existence of a non-contractible loop in this space [5]. The nontrivial structure was first demonstrated for the $SU(2)$ model with an argument that depends crucially on the fact that the symmetry is spontaneously broken. In other words, the loop disappears in the symmetric phase of the model ($M_W \to 0$). This construction therefore gives us no real information about the structure of the configuration space in the symmetric phase. The issue of this structure becomes perplexing when one tries to relate, in the weak coupling limit, the $CS = 1/2$ sphaleron S to the newly found $CS = 0$ axisymmetric S^* "new sphaleron" solution [8] which can be interpreted as a bound state of a sphaleron and an antisphaleron. More generally the structure should account for the existence of recently discovered multisphaleron bound-state solutions [9] with $CS = n/2$, $n \in \mathbf{Z}$. What remains a puzzle is the relation of the S, S^*, and multisphaleron configurations to the "bifurcated" saddle points with $CS = 1/2 \pm \epsilon(\mathrm{m}_H)$

found by Yaffe for large Higgs masses [10].

It is expected that, in the presence of large thermal fluctuations of the gauge and Higgs fields in the hot electroweak plasma, sphaleron deformations with energy equal to or higher than that of the sphaleron will contribute to rapid baryon and lepton-violating transition rates. Their presence is especially likely to dominate the symmetric high-temperature phase of the electroweak theory ($T \geq E_{sph} \sim M_W/\alpha$). In fact scaling arguments indicate unsuppressed $B+L$ violating transition rates ($\Gamma \propto T^4$) in this régime [11]. Recent computer simulations of hot electroweak sphaleron transitions in the symmetric phase [12] corroborate to this physical picture.

In two previous publications [13, 14] it was observed that the $B + L$ violating properties of the electroweak $SU(2)$ sphaleron follow directly from its odd-parity gauge-field properties. More precisely, as a consequence of the existence of $\mathbf{Z}_2$ homotopic groups of maps $[S^2/\mathbf{Z}_2, SU(2)/\mathbf{Z}_2] = \mathbf{Z}_2$ and $[S^3/\mathbf{Z}_2, SU(2)/\mathbf{Z}_2] = \mathbf{Z} \times \mathbf{Z}_2$, the Chern-Simons functional restricted to odd-parity gauge fields is a topological charge. It takes values in $\mathbf{Z}$ or in $\mathbf{Z}+1/2$ depending on whether the fields at spatial infinity have even-parity or odd-parity pure-gauge behaviour respectively. Hence the existence of the $SU(2)$ sphaleron as well as of an infinite surface of its homotopically equivalent deformations, such as loops of electroweak W string [15, 16], has a novel group-theoretic explanation.

An important implication from such a topological index for the spectrum of the three-dimensional Dirac operator is that the number of zero modes modulo 2 is a topological invariant. Indeed in the presence of an odd-parity external gauge field the number of fermionic zero modes is either 0 (mod 2) or 1 (mod 2) for fields with respectively even or odd pure-gauge behaviour at spatial infinity. We remark that the existence of $\mathbf{Z}_2$ homotopy groups is the property only of Lie groups whose centers have $\mathbf{Z}_2$ as a subgroup. These are $SU(2N)$, $SO(2N)$ and E_7 [14]. For gauge groups which do not have $\mathbf{Z}_2$ in their center the structure is more complicated.

In the present paper we demonstrate that the $\mathbf{Z}_2$ topological structure in the configuration space of a general $SU(2) \times U(1)$ model implies the existence of two topologically disconnected classes of gauge-field configurations which are odd under a generalized parity transformation, defined as the composition of an ordinary parity transformation and a gauge transformation. Eigenvalues under generalized parity are gauge invariant. The part of configuration space associated with the $\mathbf{Z}_2$ structure is infinitely large.

Furthermore we show that the $\mathbf{Z}_2$ structure accounts for the new S^* "sphaleron" with $B+L$ preserving properties [8] as well as other $B+L$ preserving configurations constructed from W-string loops and Z-string segments. It moreover provides us with sufficient conditions for the construction of infinitely many sphaleron deformations with $B+L$ violating properties, such as W-string loops, properly twisted segments of Z string connecting a monopole with an antimonopole, and multisphaleron configurations with $CS = n + 1/2$, $n \in \mathbf{Z}$. The plethora of examples of both kinds suggests that a hot electroweak plasma at sufficiently high temperature can be described qualitatively as a "two-component fluid" composed of two types of localized

excitations corresponding to configurations which permit or prohibit $B+L$ violation. The density of both types of excitations is nonzero in the symmetric phase, while the density of the $B+L$ violating component decreases rapidly to zero for $T < T_{crit}$. We speculate about the possible existence in the symmetric phase of quark-lepton condensates in the presence of such a non-zero density of $B+L$ violating gauge-Higgs field configurations.

The rest of the paper is organized as follows. In section 2 we review a topological classification for odd-parity $SU(2)$ gauge fields with pure-gauge behaviour at spatial infinity. In section 3 we extend this approach to $SU(2)$ gauge fields which are odd under a generalized parity transformation. We also generalize our approach to the $SU(2) \times U(1)$ theory. In section 4 we compute the Chern-Simons number of gauge-field configurations which are odd under generalized parity. In section 5 we construct explicit examples of sphaleron deformations using twisted loops of electroweak W string as well as Z-string [17] segments connecting a monopole with an antimonopole (Nambu strings) [18]. Single twisted loops of Z string are shown to lead to inconsistent field configurations and can exist only when linked with another vortex string. The $B + L$ preserving properties of the new S^* "sphaleron" are established, as well as the $B+L$ violating properties of multisphaleron configurations with $CS = n + 1/2$, $n \in \mathbf{Z}$. The topological $\mathbf{Z}_2$ structure implies the existence also of $B + L$ preserving multisphaleron solutions with $CS = n \in \mathbf{Z}$. We conclude with a discussion of the possible formation of fermionic condensates in the high-temperature electroweak symmetric phase in the presence of a non-zero density of localized excitations which correspond to configurations that admit an odd number of fermion zero modes.

2 An SU(2) Topological Classification

We start with the observation that the static sphaleron configuration has an odd-parity gauge field everywhere in space. While this occurs in a particular gauge it will prove sufficient for our purposes, because physical properties such as its energy and the Dirac spectrum in its background are gauge independent. By imposing parity oddness on all possible deformations (they may or may not be solutions) we find two topologically distinct sectors of configurations that depend on the (even-odd) parity property of the group-valued function that determines their pure-gauge behaviour at spatial infinity.

Let us recall some properties of the sphaleron configuration. Its gauge field reads

$$A_i = v(r)\frac{\epsilon_{ijk}x_j\tau_k}{r^2} = -iv(r)\partial_i U_{sph}\, U_{sph}^{-1}, \tag{2.1}$$

where τ_k are the Pauli matrices, $r^2 = x_j x_j$, Roman indices run over the three space components, and

$$U_{sph} = \frac{ix_k\tau_k}{r}. \tag{2.2}$$

The profile function $v(r) \to 1$ as $r \to \infty$ so that this configuration is a pure gauge at infinity.

The Chern-Simons functional is defined to be

$$CS[A] = \frac{1}{8\pi^2} \int_{D^3} \mathrm{Tr}(AdA - \frac{2i}{3}A^3) \tag{2.3}$$

and is gauge dependent. In the particular gauge defined by Eqs. (2.1) and (2.2) the Chern-Simons functional of the sphaleron configuration evaluates to zero. This fact can be checked almost without calculation due to the observation that this gauge-field configuration is odd under parity. However, the Chern-Simons *number* (CS) of the sphaleron is correctly defined as the gauge-independent difference in the value of the Chern-Simons functional when applied to the sphaleron configuration and to the vacuum. In order to make this comparison, we must transform to a gauge in which the gauge fields decrease more rapidly than $1/r$ at infinity. In such a gauge, the vacuum has zero CS number and the sphaleron has $CS = 1/2$, as we show below. The gauge transformation is accomplished by an $SU(2)$ group element U' which is smooth everywhere and coincides with U_{sph} at infinity. Since $\pi_2(SU(2)) = 0$ we know that such a field U' does exist. The Chern-Simons functional transforms gauge covariantly according to the relation

$$\mathrm{CS}[A'] = \mathrm{CS}[A] - S_{WZW}[U'], \tag{2.4}$$

where

$$A'_k = U'A_k(U')^{-1} - i\partial_k U' \, (U')^{-1}. \tag{2.5}$$

and S_{WZW} stands for the Wess-Zumino-Witten functional defined as follows

$$S_{WZW}[U'] = \frac{1}{24\pi^2} \int_{D^3} \mathrm{Tr} \, (dU' \, U'^{-1})^3. \tag{2.6}$$

This term actually depends only on the behaviour of the field U' at the boundary S^2 of the disk D^3 and is equal to $1/2$ for any field U' with the asymptotic property $U' \to U_{sph}$. For example we can take

$$U' = \exp \frac{i\pi}{2} \frac{\tau_i x_i}{\sqrt{x^2 + \rho^2}} \tag{2.7}$$

and check by an explicit calculation that $S_{WZW}[U'] = -1/2$. In turn $\mathrm{CS}[A] = 0$ since the field A is odd under parity, i.e. $A_k(-x) = -A_k(x)$. Thus we conclude that $\mathrm{CS}[A'] = 1/2$, which defines the CS number of the sphaleron. It is now clear that the same value of the Chern-Simons number corresponds to all odd-parity configurations which behave as U_{sph} at infinity.

We see that the odd parity of the spherical sphaleron is essential in order for the Chern-Simons number of the sphaleron to be exactly $1/2$. However, this is not sufficient by itself. Actually, as we shall see below, the half-integer value of the CS number also depends crucially on the odd parity of U' at infinity. In a previous publication [13] a connection was established between the Chern-Simons number

of gauge fields and the parity property of their pure-gauge behaviour (i.e. the U' field itself) at infinity. We argued there that a restriction to odd-parity gauge-field configurations allows us to introduce a useful topological classification among these fields.

For gauge fields which are pure gauge at infinity (i.e. on the boundary of a 3-dimensional ball, which is S^2)

$$A_i = -i(\partial_i U)\, U^{-1}, \tag{2.8}$$

where U belongs to the $SU(2)$ group. The restriction of this field U to the boundary S^2 of the 3-dimensional ball is a map of S^2 into $SU(2)$. The homotopic group $\pi_2(SU(2))$ is trivial and hence all such configurations in the 3-dimensional ball are contractible to unity. We now restrict ourselves to the space of 3-dimensional odd-parity gauge fields. We shall argue that there exists a relevant non-trivial homotopic classification in this space of gauge fields. Indeed let us consider an odd-parity configuration

$$A_i(-x) = -A_i(x). \tag{2.9}$$

On the S^2 boundary this is a pure gauge so that

$$(\partial_i U)(-x)U^{-1}(-x) = -(\partial_i U)(x)U^{-1}(x), \tag{2.10}$$

or equivalently

$$\partial_i\left[U^{-1}(-x)U(x)\right] = 0. \tag{2.11}$$

From this it follows that the matrix $V = U^{-1}(-x)U(x)$ is a non-degenerate constant $SU(2)$ element, and consequently the equation

$$U(x) = U(-x)V \tag{2.12}$$

is valid for any point x on S^2. From this equation, combined with that obtained by changing $x \to -x$, we find

$$V^2 = 1. \tag{2.13}$$

A short inspection now shows that the matrix V should belong to the center of $SU(2)$. In equations

$$V = \pm 1. \tag{2.14}$$

Thus we get two different classes of odd-parity gauge fields: Those with odd U and those with even U.

This conclusion reflects a non-triviality of the homotopy group of maps from the projective sphere $\mathbf{R}P^2 = S^2/\mathbf{Z}_2$ (where $\mathbf{Z}_2$ is a group of parity transformations with respect to some point in 3-dimensional space) to the group $SO(3) = SU(2)/\mathbf{Z}_2$ (where $\mathbf{Z}_2$ is the center of $SU(2)$). In short $[\mathbf{R}P^2, SO(3)] = \mathbf{Z}_2$. The consideration above shows that the odd-parity gauge fields split into two topologically disconnected equivalence classes. In other words it is not possible to get from one to the other continuously through odd-parity gauge-field configurations.

Actually the classification of these gauge fields is more complicated. Instead of restricting to the sphere S^2, we must take into consideration the set of gauge-field configurations $A_i(x)$ which are odd under parity in the entire 3-dimensional ball. The relevant homotopy group is then $[\mathbf{R}P^3, SO(3)] = \mathbf{Z} \times \mathbf{Z}_2$. [5] The factor $\mathbf{Z}_2$ can be identified with the above classification in terms of odd U or even U. Each of these two classes of odd-parity fields thus possesses its own $\mathbf{Z}$ structure of topologically disconnected components. For fields with even U this structure is directly related to the periodic structure of the Yang-Mills vacuum. More precisely, each component contains one of the vacua $A_i^n = -i(\partial_i U_n)U_n^{-1}$, where the group element U_n is given by the even-parity (at infinity) group elements [19]

$$U_n = \exp(in\pi \frac{\tau_i x_i}{\sqrt{x^2 + \rho^2}}). \tag{2.15}$$

Here ρ is a constant parameter and n is the integer Chern-Simons number of a vacuum configuration. Because there is no additional structure, it follows that any configuration with an even U field at infinity is continuously connected to one of these vacua.

As we saw above, the sphaleron gauge field is (in an appropriate gauge) odd under parity and has an odd U (see Eq. (2.2)) and hence it is disconnected from the configurations with even U. The $\mathbf{Z}$ structure of the fields with odd U at infinity implies the existence of an infinite sequence of topologically distinct sphaleron solutions associated with saddle points on the energy barriers separating neighboring vacua. Because there is no additional structure, any configuration with an odd U field at infinity will be continuously connected to one of these sphalerons.

A common feature of all odd-parity, even-U configurations is that they have an integer-valued Chern-Simons number. Indeed, in analogy with the case of the sphaleron we can make a nonsingular gauge rotation which removes the gauge field at infinity. The Wess-Zumino-Witten functional would then give us the Chern-Simons number of the gauge-field configuration. It is easy to see that the value of $S_{WZW}[U]$ is invariant under even-parity smooth deformations of the even U field at infinity. Indeed a variation of the Wess-Zumino-Witten functional reads

$$\delta S_{WZW}[U] = \frac{1}{8\pi^2} \int_{D^3} d\mathrm{Tr}((U^{-1}\delta U)(U^{-1}dU)^2) = \frac{1}{8\pi^2} \int_{S^2} \mathrm{Tr}(U^{-1}\delta U)(U^{-1}dU)^2. \tag{2.16}$$

Since the variation of the group element on the surface $\partial D^3 = S^2$ is even under parity and its value depends only on the values of the fields at the boundary we immediately conclude that the present variation of the Wess-Zumino-Witten functional equals zero. On the other hand, let us consider a product of even-parity (at infinity) group elements U_1 and U_2. It is easy to check that

$$S_{WZW}[U_1 U_2] = S_{WZW}[U_1] + S_{WZW}[U_2] + \frac{1}{8\pi^2} \int_{D^3} d\, \mathrm{Tr}\left((U_1^{-1}dU_1)(dU_2\, U_2^{-1})\right). \tag{2.17}$$

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[5]Notice that this statement is not quite rigorous from the mathematical point of view. Actually as it is shown in Ref. [13] the correct formulation is $[\mathrm{R}P^3, SO(3)] = \mathbf{Z}$ where $\mathbf{Z}$ is doubly covered. Such a double covering is induced by the map $SU(2) \to SO(3) = SU(2)/\mathbf{Z}_2$.

The third term in the right-hand side of this equation equals zero due to the odd parity of the integrand at infinity. Hence we see that the Wess-Zumino-Witten functional acts as a homomorphism from the group of maps U to a discrete subgroup of the group of real numbers which is obviously isomorphic to $\mathbf{Z}$. As we argued before, any even-U (at infinity) group element is contractible to a vacuum. On the other hand, as it is well known that this vacuum can have any integer value of the Chern-Simons number, we conclude that even-parity U fields are indeed classified by $\mathbf{Z}$.

Thus all odd-parity even-U gauge fields split into an infinite set of disconnected equivalence classes which are labeled by integer values of their Chern-Simons numbers.

Let us now consider odd-parity odd-U gauge fields. A similar argument shows that the value of the Chern-Simons functional is a topological invariant while the Wess-Zumino-Witten functional maps the odd-parity U fields to a discrete subgroup of the group of real numbers according to Eq. (2.17). On the other hand a product of two odd-parity group elements U_1 and U_2 is even under parity. By taking into account that the sphaleron has $S_{WZW}[U] = -1/2$ we conclude that the odd-parity odd-U gauge fields have half-integer values of the Wess-Zumino-Witten functional and hence are classified by $n + 1/2$ ($n \in \mathbf{Z}$), while these equivalence classes are themselves topologically disconnected from each other for different values of n.

Thus we see that the Chern-Simons number plays the role of a topological charge: It takes values in $\mathbf{Z}$ for even-U and in $\mathbf{Z} + 1/2$ for odd-U (at infinity) fields respectively.

An immediate implication is that the Chern-Simons number acts as a topological index for the spectrum of the 3-dimensional Dirac operator. Let us consider a Dirac operator $\slashed{D} = \sigma_i(\partial_i - iA_i)$ in an external odd-parity gauge field A_i. It is easy to see that the non-zero eigenvalues appear in pairs $(\lambda, -\lambda)$. Indeed if $\psi(x)$ is a wave function corresponding to an eigenvalue λ then $\psi(-x)$ is an eigenfunction corresponding to an eigenvalue $-\lambda$. Hence, when the external field varies continuously, the number of zero modes of the Dirac operator is invariant modulo 2. For the sphaleron background this topological invariant is equal to one while for the vacuum its value is zero.

Since odd-U configurations are continuously connected to a sphaleron, and even-U configurations to a vacuum, the invariance modulo 2 of the number of zero modes implies that the number of fermionic zero modes in an odd-parity background gauge field is 0 mod 2 for even-U and 1 mod 2 for odd-U configurations.

3   A Generalized Parity Symmetry

In this section we extend our topological classification to gauge fields which are non-odd under a parity transformation.

Let us consider a semi-simple group G. We define generalized parity to be the parity up to a gauge transformation. Let us consider gauge fields odd under generalized parity, i.e. fields that satisfy the equation

$$\begin{aligned} A_i(-x) = -A_i^S(x) &\equiv -S(x)(i\partial_i + A_i(x))S^{-1}(x) \\ &= -S(x)A_i(x)S^{-1}(x) + i\partial_i S(x)S^{-1}(x), \end{aligned} \tag{3.1}$$

for some $S(x) \in G$. As a trivial example, consider an ordinary odd-parity gauge field which has been transformed with an arbitrary parameter of the gauge group. The transformed field is then odd under generalized parity. On the other hand, if we pick a gauge field which is odd under generalized parity, one can show that it is still odd under generalized parity after an arbitrary gauge transformation. Therefore, generalized parity symmetry is a gauge-invariant property. However, as we shall show below, the set of gauge fields odd under generalized parity is larger than the set of gauge fields odd under ordinary parity. For the continued discussion, we consider the gauge groups $SU(2)$ and $SU(2) \times U(1)$ separately.

3.1 The Case of SU(2) Generalized Parity

Consider a field $A_i(x)$ which is odd under the generalized parity transformation defined by $S(x) \in SU(2)$. Then

$$F_i(-x) = S(x)F_i(x)S^{-1}(x), \tag{3.2}$$

where

$$F_i = \frac{1}{2}\epsilon_{ijk}F_{jk}, \quad i = 1, 2, 3, \tag{3.3}$$

and F_{jk} is the strength of the gauge field. Combining Eq. (3.2) with the same equation for $x \to -x$, one obtains

$$F_i(x) = \Omega(x)F_i(x)\Omega^{-1}(x), \tag{3.4}$$

where

$$\Omega(x) = S(-x)S(x). \tag{3.5}$$

From Eq. (3.4) one can see that Ω is diagonalized simultaneously with three components F_i. There are two possibilities:

i) $\Omega = \pm 1$,

ii) F_i gets factorized as follows

$$F_i(x) = f_i(x) \cdot i\Phi(x), \tag{3.6}$$

where f_i are scalar functions, $\Phi(x)$ belongs to the $SU(2)$ algebra, and

$$\Omega(x) = \exp(i\lambda(x)\Phi(x)). \tag{3.7}$$

8

Here $\lambda(x)$ is a scalar function.

In the second case the gauge field A_i belongs to an abelian subgroup of $SU(2)$, and hence can be taken to be proportional to the τ_3 generator of the $SU(2)$ algebra. In what follows we focus on the first case, which more interesting because it corresponds to non-abelian gauge fields.

Let us find a necessary condition for the gauge field A_i that obeys Eq. (3.1) not to be gauge equivalent to a gauge field $\tilde{A}_i$ which is odd under the ordinary parity transformation (i.e. $\tilde{A}_i(-x) = -\tilde{A}_i(x)$). In terms of the field strength, such a reducibility would imply that we can find a gauge transformation T such that

$$\tilde{F}_i(x) = T(x)F_i(x)T^{-1}(x) \tag{3.8}$$

where $\tilde{F}$ is even under the usual parity transformation,

$$\tilde{F}_i(-x) = \tilde{F}_i(x). \tag{3.9}$$

From these equations and Eq. (3.2) it follows that

$$T(-x)S(x)T^{-1}(x) = \pm 1 \tag{3.10}$$

or, equivalently,

$$S(x) = \pm T^{-1}(-x)T(x). \tag{3.11}$$

Combining this equation with that obtained by taking $x \to -x$ we easily get

$$\Omega(x) = 1. \tag{3.12}$$

In this paper we concentrate on the simplest case of gauge fields which are invariant under a generalized parity transformations with $S(x) \equiv B = const$. Therefore let us apply the above analysis to this special case. One easily gets from the above equations that

$$B^2 = 1,$$

and hence (since $B \in SU(2)$)

$$S(x) = B = \pm 1, \tag{3.13}$$

which implies that the gauge field $A_i(x)$ is odd under an *ordinary* parity transformation. Therefore we see that there exist gauge field configurations invariant under generalized parity with a constant S which are not reducible to gauge configurations which are invariant under an ordinary parity transformation. Thus the notion of a generalized parity is not empty. In section 5 we shall give explicit examples of such configurations.

We shall now proceed by showing that, for $SU(2)$ gauge fields with odd generalized parity, there exists a non-trivial homotopic classification in terms of the parity properties of the $SU(2)$-valued function $U(x)$ which determines the pure gauge behavior

$$A_i(x) \sim -i(\partial_i U U^{-1})(x) \tag{3.14}$$

9

at infinity. By using eq.(3.1) at spatial infinity we find that

$$(\partial_i UU^{-1})(-x) = -B(\partial_i UU^{-1})(x)B^{-1}, \tag{3.15}$$

and therefore

$$\partial_i(U^{-1}(-x)BU(x)) = 0. \tag{3.16}$$

At infinity we must therefore have that

$$BU(x) = U(-x)V, \tag{3.17}$$

where $V \in SU(2)$ is a constant matrix. Using this equation and its parity conjugate one easily gets

$$B^2U(x) = U(x)V^2, \tag{3.18}$$

and thus

$$U(x)^{-1}B^2U(x) = V^2 \in SU(2). \tag{3.19}$$

Let us analyze the different cases for the matrix B^2. Let us first assume that $B^2 \neq \pm 1$. Then from Eq. (3.19) we get

$$U(x)^{-1}BU(x) = \pm V. \tag{3.20}$$

Combining Eq. (3.20) with Eq. (3.17) we obtain

$$U(-x) = \pm U(x), \tag{3.21}$$

from which it can be checked that $B^2 = \pm 1$ in contradiction of our assumption. Therefore we conclude that $B^2 = V^2 \in \mathbf{Z}_2$. Then there are two possible cases.

1. If $B^2 = 1$ then $B = \pm 1$. Then $U(-x) = \pm U(x)$, which is the case of odd or even U found already in section 2.

2. If $B^2 = -1$ then B belongs to the coajoint orbit $\mathcal{B} = SU(2)/U(1)$, i.e.

$$B \in \mathcal{B} = \{iS^\dagger\sigma_3 S | S \in SU(2)\}. \tag{3.22}$$

This orbit is isomorphic to an S^2 sphere.

Thus we conclude that there exist the following two classes:

$$\mathbf{1.} \quad U(x) = \pm U(-x), \tag{3.23}$$

$$\mathbf{2.} \quad \tau U(x) = \pm U(-x)\tau', \quad i\tau, \ i\tau' \in \mathcal{B}. \tag{3.24}$$

We see that the condition of class 2 can be satisfied by a constant matrix $U(x) = const$. This class is then topologically trivial, because it intersects with the class of U-even configurations $U(x) = U(-x)$ which contains the vacuum. Thus the topologically non-trivial configurations satisfy $U(-x) = -U(x)$ and belong to class 1, which has been analyzed in the previous section and in Ref. [13].

3.2 The Case of SU(2)×U(1) Generalized Parity

Let us now consider the case of an $SU(2) \times U(1)$ gauge theory. We focus on the gauge fields which obey the following condition (a generalized parity)

$$A(-x) = -BA(x)B^\dagger, \tag{3.25}$$

where B is a constant matrix belonging to $SU(2)/\mathbf{Z}_2$, $\mathbf{Z}_2 = \pm 1$ is the center of $SU(2)$, and $B \neq \pm 1$.

At infinity we have

$$A_i(x) = -i(\partial_i U U^{-1})(x), \tag{3.26}$$

where $U(x) \in U(2)$. Hence

$$(\partial_i U U^{-1})(-x) = -B(\partial_i U U^{-1})(x)B^\dagger. \tag{3.27}$$

Therefore we conclude that

$$BU(x) = U(-x)V, \tag{3.28}$$

where $V \in U(2)$ is constant matrix.

Using this equation twice we get

$$B^2 U(x) = U(x)V^2, \tag{3.29}$$

and thus

$$U(x)^{-1}B^2 U(x) = V^2 \in SU(2). \tag{3.30}$$

We may classify the different cases for the matrix B^2:

1. $B^2 \in \mathbf{Z}_2$, i.e. $B^2 = \pm 1$. Then $V^2 \in \mathbf{Z}_2$ and $B^2 = V^2$. Since B does not belong to $\mathbf{Z}_2$ (the center of $SU(2)$) we have two subcases

(1a). $B = iS^\dagger \sigma_3 S$, $V = \pm i$ or $V = S'^\dagger i \sigma_3 S'$, where $S, S' \in SU(2)$.

(1b). $B = S^\dagger \sigma_3 S$, $V = \pm 1$ or $V = S'^\dagger \sigma_3 S'$, where $S, S' \in SU(2)$.

2. $B^2 \neq \pm 1$. Then from Eq. (3.30) we get

$$U(x)^{-1}BU(x) = \pm V. \tag{3.31}$$

Substituting Eq. (3.31) into Eq. (3.28) we come to the condition $U(-x) = \pm U(x)$, i.e. to the old case of odd and even $U(x)$.

As a result we have the following classification of pure-gauge behaviour of the gauge fields odd under a generalized parity defined in Eq. (3.25) (let us denote this subspace of the space of the gauge fields as $\mathcal{A}^P$) :

$$\mathbf{1.} \quad \tau U(x) = \pm U(-x), \quad i\tau \in \mathcal{B}, \tag{3.32}$$

$$\mathbf{2.} \quad \tau U(x) = \pm U(-x)\tau', \quad i\tau, i\tau' \in \mathcal{B}, \tag{3.33}$$

$$\mathbf{3.} \quad U(x) = \pm U(-x)\tau, \quad i\tau \in \mathcal{B}, \tag{3.34}$$

$$\mathbf{4.} \quad U(x) = \pm U(-x). \tag{3.35}$$

11

Now we can look at each of the above four classes. Class 4 (Eq. (3.35)) corresponds to gauge fields odd under the ordinary parity transformation. For the case of an $SU(2)$ gauge group such fields have been classified in Ref. [13]. There it was shown that the gauge fields with pure-gauge behaviour at infinity corresponding to an even-parity group element are topologically trivial and have integer Chern-Simons number. The gauge fields with odd-parity pure-gauge behaviour at infinity are topologically non-trivial and have half-integer Chern-Simons number. Here we see that an extension of that analysis to the case of the $SU(2) \times U(1)$ group is straightforward; the classification of odd-parity gauge fields is the same as with the case of the $SU(2)$ group.

Let us now look at the other cases. Class 2. (Eq. (3.33)) can be satisfied by a constant matrix $U(x) = const$. Hence it is topologically trivial since it intersects with the class of fields obeying the condition $U(-x) = U(x)$ such as the trivial vacuum.

Classes 1. and 3. (eqs. (3.32) and (3.34) respectively) do not contain any constant matrices and could hence *a priori* correspond to topologically non-trivial configurations (i.e. which would allow for fermion level crossing). However, it is easy to see that these classes are empty. It readily follows from the trivial fact that $\pi_2(S^1) = 0$.

3.3 Spectrum of the Dirac Operator

We now consider the Dirac equation

$$\sigma_i(\partial_i - iA_i(x))\psi(x) = i\lambda\psi(x) \tag{3.36}$$

in the case of an external gauge field obeying Eq. (3.1) with $S(x) \in SU(2) \times U(1)$. We can show that there is a pairing up of non-zero levels of the Dirac operator. To that effect we make a parity reflection of the Dirac equation Eq. (3.36). It is easy to see by using Eq. (3.1) that

$$\sigma_i S(x)(\partial_i - iA_i(x))(S^{-1}(x)\psi(-x)) = -i\lambda\psi(-x). \tag{3.37}$$

Thus we see that the wave function

$$\psi^P(x) = S^{-1}(x)\psi(-x) \tag{3.38}$$

corresponds to an eigenvalue $-\lambda$. Therefore if one deforms an external gauge field among the fields obeying Eq. (3.1) for some $S(x)$ then the number of zero modes of the Dirac operator is a topological invariant modulo 2. Hence the fields obeying Eq. (3.1) split into two classes: in the first one the Dirac operator has an even number of zero modes and in the second one it has an odd number of zero modes. The first class is topologically trivial since it contains the vacuum configuration. The second one contains the sphaleron configuration and is non-trivial in the sense that any configuration that belongs to it cannot be continuously deformed into a vacuum configuration as long as we keep Eq. (3.1) satisfied.

Thus if we compare the present general cases to that considered in Ref. [13] (for the gauge field odd under the conventional parity reflection) we can see that we still have two topologically distinct classes while these classes are enlarged as compared to those of Ref. [13].

This analysis actually gives the following simple interpretation of the above classification of the gauge fields with definite generalized parity. The subspace in the space of the gauge fields which corresponds to those with definite generalized parity can be continuously constructed starting with the gauge fields with definite ordinary parity. Indeed the latter correspond to the case of the group element $S = 1$ in Eq. (3.1). They are separated into two disconnected homotopy classes. Let us now continuously deform the matrix S away from 1. By such a procedure we extend our homotopy classes to two disconnected varieties, which are infinitely larger than the classes of gauge fields with definite ordinary parity. The fact that these two varieties are larger than those for $S = 1$ was actually proven in section 3.1 above, where it was shown that there exist gauge field configurations which are invariant under a generalized parity transformations and are not gauge equivalent to those which are invariant under the ordinary parity transformations. Some explicit examples of such nontrivial deformations given in section 5 support this statement.

Moreover as follows from the above analysis of the spectrum of the Dirac operator the number of zero modes of the Dirac operator remains a topological invariant modulo 2. That means that the two varieties which are constructed by deformations of the group element S remain homotopically disconnected.

An important question is now if the enlarged classes contain gauge fields with the Chern-Simons numbers different from 0 and 1/2 (modulo an integer). We address this question in next section.

4   The Chern-Simons Functional

Let us first calculate the value of the Chern-Simons functional for a gauge field obeying Eq. (3.1). This functional is defined as follows

$$CS[A] = \frac{1}{8\pi^2} \int_{D^3} \text{Tr}(AdA - i\frac{2}{3}A^3). \tag{4.1}$$

For odd-parity configurations $CS = 0$. For the fields odd under the generalized parity we have

$$CS[A] = \frac{1}{8\pi^2} \int_{D^3} \text{Tr}(A(x)dA(x) - i\frac{2}{3}A^3(x)) = \tag{4.2}$$

$$= -\frac{1}{8\pi^2} \int_{D^3} \text{Tr}(A(-x)dA(-x) + i\frac{2}{3}A^3(-x)) =$$

$$= -\frac{1}{8\pi^2} \int_{D^3} \text{Tr}(A^S(x)dA^S(x) - i\frac{2}{3}(A^S)^3(x)) = -CS[A^S],$$

where

$$A^S = S(x)(i\partial_i + A_i(x))S^{-1}(x). \tag{4.3}$$

On the other hand we have the identity

$$CS[A^S] = CS[A] - S_{WZW}[S], \tag{4.4}$$

where

$$S_{WZW}[S] = \frac{1}{24\pi^2} \int_{D^3} \text{Tr}(dSS^{-1})^3 \tag{4.5}$$

and we assume that $S^{-1}dS \sim 1/|x|^{1+\epsilon}$ at infinity ($\epsilon > 0$). Substituting this identity (4.4) into Eq. (4.2) we get

$$CS[A] = \frac{1}{2}S_{WZW}[S]. \tag{4.6}$$

One can see that this value reduces to zero for any gauge field A_i odd under the ordinary parity reflection since in this case $S = 1$.

However the value of the Chern-Simons functional (4.6) is not yet associated with a given gauge field. To define correctly the Chern-Simons number we have to make a gauge transformation with an appropriate parameter $U^\dagger$ in order to remove field at infinity, and then evaluate the Chern-Simons functional for the gauge transformed field $W = A^{U^+}$. The field W is assumed to decrease rapidly like $\sim 1/|x|^3$ at infinity. Using Eq. (4.4) one can get

$$CS[W] = CS[A] + S_{WZW}[U] = \frac{1}{2}S_{WZW}[S] + S_{WZW}[U]. \tag{4.7}$$

For group elements $U(x)$ which are odd (even) at infinity, whose corresponding gauge fields are odd under the ordinary parity transformation, $S_{WZW} = 1/2$ ($S_{WZW} = 0$) modulo an integer. For gauge fields odd under the generalized parity both $S_{WZW}[S]$ and $S_{WZW}[U]$ may be fractional. Thus one could expect that the value of the Chern-Simons functional can assume any fractional number when we enlarge the topologically distinct classes. In what follows we will show that this is not the case. First, observe that the group elements U and S are related by the following condition at infinity which can be deduced from Eq. (3.1)

$$(dUU^{-1})(x) = -(d(SU)U^{-1}S^{-1})(-x). \tag{4.8}$$

Let us calculate a variation of $CS[W]$ under a small deformation of $S(x)$. The variation of the Wess-Zumino-Witten functional is given by eq.(2.17). Thus we see that the variation depends only on the behaviour of the group element at infinity. Changing $x \to -x$ under the integral and using Eq. (4.8) one can easily get

$$\delta S_{WZW}[U] = -\delta S_{WZW}[U] - \delta S_{WZW}[S], \tag{4.9}$$

and hence

$$\delta S_{WZW}[U] + \frac{1}{2}\delta S_{WZW}[S] = 0. \tag{4.10}$$

14

Thus we see that $CS[W]$ is invariant under continuous deformations of the group element $S(x)$. Therefore the only available values of the Chern-Simons functional within our two topological classes are 0 and 1/2 modulo an integer.

We conclude that the space of gauge fields odd under a generalized parity has a $\mathbf{Z}_2$ structure similar to the space of the odd-parity fields.

In the previous section we have shown that the Wess-Zumino-Witten functional realizes a homomorphism into $\mathbf{Z}_2$ from the set of the $SU(2)$ group valued functions corresponding to ordinary odd-parity gauge fields A_i. Such a homomorphism maps a product of two $SU(2)$ group-valued functions into a sum of the values of the Wess-Zumino-Witten functionals for each of them. In the case of a generalized parity we do not have such a homomorphism because a product of two group elements corresponding to a pure-gauge behaviour at infinity is not in general an element of the same sort.

5 A Multitude of Sphaleron Deformations

We now proceed to give concrete examples which illustrate the $\mathbf{Z}_2$ structure in the space of gauge field configurations. The purpose is to demonstrate that there is an infinity of background configurations other than the sphaleron which admit an odd number of fermion zero modes and therefore contribute to baryon-number changing processes in the hot electroweak theory. The configurations must approach pure gauge at infinity and we take them to be of the form

$$A_i(x) = -iv(x)(\partial_i U(x))U^{-1}(x), \tag{5.1}$$

$$\Phi(x) = f(x)\, U(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5.2}$$

where v and f are profile functions with the property that $v, f \to 1$ at spatial infinity.

It is important to realize that the spectrum of the Dirac equation (neglecting Yukawa couplings) depends only on the background gauge fields, and therefore the number of Dirac zero modes in a given gauge background will be the same in the broken phase as in the hot symmetric phase of the theory, in which the Higgs-field expectation value is zero. We thus obtain a means of understanding the mechanism of baryon-number changing transitions in the symmetric phase as well as in the broken phase. The effect of Yukawa couplings in the broken phase is merely to deform slightly the set of configurations for which fermionic zero modes occur (see Ref. [20]) without reducing the number of such configurations.

In the symmetric phase, the energy density of the (static) gauge configuration (5.1) is $\operatorname{Tr} F^2/(4g^2)$, where

$$F_{kl} = -i(\partial_{[k}v)(\partial_{l]}U)U^{-1} + iv(1-v)(\partial_{[k}U)(\partial_{l]}U^{-1}) \tag{5.3}$$
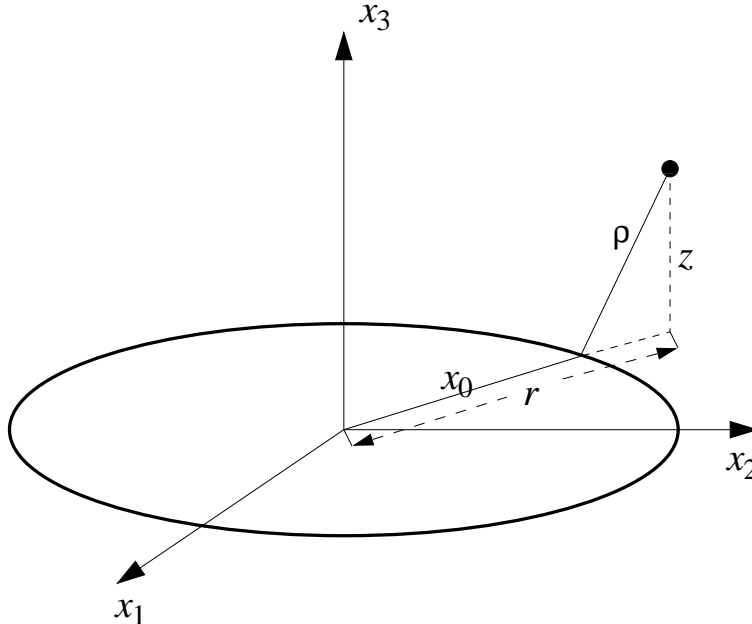
Figure 1: *Definition of cylindrical coordinates for circular loop configurations.*

and $A_{[k}B_{l]} \equiv A_k B_l - A_l B_k$. The energy density will be singular at points where the group element U is multivalued. If the set of singular points constitutes a one-dimensional string, it is easy to show that integrability of the energy density requires the behavior $v \sim \rho^k$, $k > 1$ of the profile function near the string, where ρ is the perpendicular distance to the string. It follows that the gauge-field configuration is, also in the symmetric phase, that of an embedded Nielsen-Olesen vortex [16, 21]. On the other hand, if the group element U is multivalued at a finite number of points, these point singularities may be suppressed simply by providing appropriate isolated zeros of the profile function v.

Let us apply our classification of gauge fields, developed in section 3, to Nielsen-Olesen vortex-like deformations of the sphaleron, using the circular string loop given by

$$A_i = -iv(\rho)\partial_i UU^{-1}, \tag{5.4}$$

where $\rho = \sqrt{(r - x_0)^2 + z^2}$ and r, z are cylindrical coordinates. This loop has radius x_0 and resides in the plane $z = 0$ with its center at $r = z = 0$ (see Fig. 1). The profile function v here satisfies $v(0) = 0$, $v(\infty) = 1$, and its zeros define the center of the vortex string.

Let us first consider loops of W-string [15, 16]. The W-string is a Nielsen-Olesen vortex [22] embedded in the electroweak theory and is characterized by an $SU(2)$ vector potential which wraps around the string. Loops of this string have been

studied in Ref. [13]. The simplest loop configuration corresponds to the $SU(2)$ group element

$$U_W = \frac{1}{\sqrt{(r-x_0)^2 + z^2}} \begin{pmatrix} r - x_0 & z \\ -z & r - x_0 \end{pmatrix}. \tag{5.5}$$

More generally, one can apply a "twist" which depends on the azimuthal angle ϕ and changes as one goes along the loop. Consider first the group elements

$$U = \exp(i(2n+1)\phi\tau_3/2)\frac{1}{\sqrt{(r-x_0)^2 + z^2}} \begin{pmatrix} r - x_0 & z \\ -z & r - x_0 \end{pmatrix} \exp(i(2m+1)\phi\tau_3/2), \tag{5.6}$$

where n, m are integers. These group elements have a definite eigenvalue under the ordinary parity transformation. We find that for odd $m + n$ the matrix U is even under parity, and the corresponding configurations are topologically trivial (see section 2) with integer Chern-Simons number. In contrast, the configurations with even $m + n$ correspond to odd U matrices and are topologically non-trivial with half-integer CS number. Therefore, the configurations given by Eq. (5.6) for any x_0, m, n and even $m + n$ admit an odd number of fermion zero modes and are as relevant as the sphaleron for baryon-number changing processes at sufficiently high temperature.

The topologically non-trivial configurations are deformations of the sphaleron, as can be seen from the following example. Consider the symmetrically twisted loop with $m = n = 0$ and take the limit of a collapsed loop $(x_0 \to 0)$. We obtain

$$\begin{aligned} U \quad &\to \quad \exp(i\phi\tau_3/2)\frac{1}{\sqrt{r^2 + z^2}} \begin{pmatrix} r & z \\ -z & r \end{pmatrix} \exp(i\phi\tau_3/2) \\ &= \quad \frac{1}{\sqrt{r^2 + z^2}} \begin{pmatrix} x + iy & z \\ -z & x - iy \end{pmatrix} = -i\tau_2 U_{sph}\, i\tau_3 \;, \end{aligned} \tag{5.7}$$

where U_{sph} corresponds to the sphaleron configuration of Eq. (2.2). Thus the collapsed loop coincides (up to a profile function) with the sphaleron configuration.

For the particular values $n = 0$, $m = -1$ one obtains an asymmetrically twisted, topologically trivial, loop which can be interpreted as a bound state of two sphalerons [13]. In this case the $SU(2)$ group element can be shown explicitly to be continuously connected to $U = 1$.

We now turn to examples of gauge-field configurations with a definite generalized parity (see section 3). Consider the twisted loops corresponding to

$$U = \exp(im\phi\tau_3)\frac{1}{\sqrt{(r-x_0)^2 + z^2}} \begin{pmatrix} r - x_0 & z \\ -z & r - x_0 \end{pmatrix} \exp(in\phi\tau_3), \tag{5.8}$$

where m and n are integers. This matrix corresponds to a gauge field which is odd under a generalized parity transformation and obviously belongs to class 2 in the classification of section 3. Hence the corresponding gauge field of the twisted loop is topologically trivial.

In Ref. [13] it was shown that the configuration given by Eq. (5.6) for general n and m can be deformed, in a process referred to as "splitting", into a configuration which corresponds to a collection of separated simple fundamental ($m = n = 0$ and $m = -1$, $n = 0$) loops. That analysis can be easily generalized to the $SU(2) \times U(1)$ gauge theory and to gauge-field configurations which have a definite generalized parity.

Let us now investigate the possibility of W-string loops with an $SU(2) \times U(1)$ twist. They would correspond to group elements U_L^{mn} or U_R^{mn} given by

$$U_L^{mn} = \exp(im\phi\tau_3) \exp(ik\phi\frac{1+\tau_3}{2}) \frac{1}{\sqrt{(r-x_0)^2 + z^2}} \begin{pmatrix} r - x_0 & z \\ -z & r - x_0 \end{pmatrix} \exp(in\phi\tau_3)$$

(5.9)

and

$$U_R^{mn} = \frac{1}{\sqrt{(r-x_0)^2 + z^2}} \exp(im\phi\tau_3) \begin{pmatrix} r - x_0 & z \\ -z & r - x_0 \end{pmatrix} \exp(ik\phi\frac{1+\tau_3}{2}) \exp(in\phi\tau_3).$$

(5.10)

where k, m, and n are integers. These configurations have been considered previously [23] in the special case $m = n = 0$.

Although the matrices U_L^{mn} and U_R^{mn} fit into our classification of gauge-field configurations with definite generalized parity, we shall show below that they correspond to configurations of infinite energy and will be physically allowed only if one introduces at least one additional vortex loop linked with the original one. This is because their twists as a function of the angle ϕ are singular on the line $r = 0$. In fact, any $U(1)$ twist or a twist with a $U(1)$ component will require such an additional loop. In contrast, in the case of the $SU(2)$ gauge group the singularity at $r = 0$ can be easily eliminated without the need for extra loops. To prove these statements we proceed as follows.

Consider a single closed vortex loop C_1 of any shape (C_1 being the set of zeros of v), subject to the condition that it does not link with itself (i.e. is not knotted). This means that $L(C_1, C_1) = 0$, where the linking number of two arbitrary curves C_1 and C_2 is defined by [24]

$$L(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2 \times (\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3}.$$

(5.11)

We first assume that the group element U is single-valued everywhere except on C_1. Consider then a second closed curve $\gamma_1(\epsilon)$ following C_1 at a distance ϵ (see Fig. 2) but not linking with it, i.e. $L(C_1, \gamma_1(\epsilon)) = 0$. The assignment of group elements U to points on the curve $\gamma_1(\epsilon)$ is, topologically, a map from a circle S^1 into the group. If the group is $U(1)$ (which is topologically equivalent to S^1), such a map is characterized by an integer topological winding number which measures the number of times U runs around $U(1)$ as one traverses the closed loop $\gamma_1(\epsilon)$. Since $\gamma_1(\epsilon)$ does not link with C_1, it may be contracted within $R^3 \setminus C_1$ to a point P along any surface bounded by $\gamma_1(\epsilon)$. If the winding number of the map is nonzero, the image of the
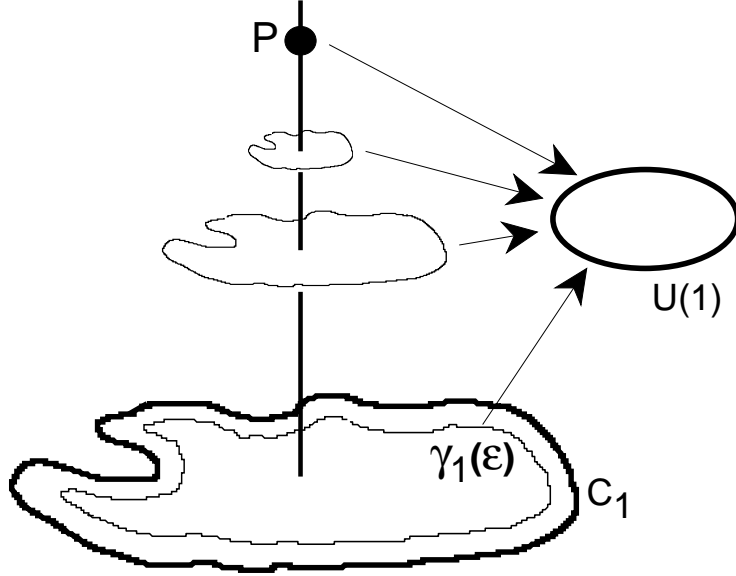
Figure 2: *A single closed vortex loop C_1 accompanied by a test path $\gamma_1(\epsilon)$ tracing it at a distance ϵ. If the map $\gamma_1(\epsilon) \mapsto U(1)$ has non-zero winding number, and if it is continuous as $\gamma_1(\epsilon)$ is shrunk to a point P, then the map is multivalued at P.*

map cannot be contracted within $U(1)$ and the group element U will be multivalued at P. By smoothly varying the surface along which $\gamma_1(\epsilon)$ is contracted, the set of points P forms a one-dimensional curve along which U is multi-valued. The ensuing singularity must then be compensated by a zero in the profile function v, and we are presented with another vortex C_2 which passes through the loop C_1. Because we require pure gauge behavior at spatial infinity in all directions, C_2 must be a closed loop which links with C_1. A similar argument, valid for $U(1)$ and configurations with $U(1)$ factors, was arrived at independently in Ref. [25].

In the case of larger gauge groups, such as $SU(2)$ which is isomorphic to the sphere S^3, the image of the map from $\gamma_1(\epsilon)$ to the group will be a closed curve embedded in the simply connected group manifold M of dimension $d > 1$, and this curve can be easily contracted within M to a unique element as $\gamma_1(\epsilon)$ is contracted to P. Therefore, U will be single-valued everywhere except on C_1, and there is no need to introduce another vortex loop. The singularity in the $SU(2)$ configurations (5.6) and (5.8) arises because their particular form requires that U take values in a one-dimensional subset of $M = SU(2)$ as $r \to 0$ for constant z.

Let us now give examples of regularized versions of the SU(2) configurations. Consider first the simplest non-trivial loop with the U matrix given in Eq. (5.6) with $m = n = 0$. We want to write down an $SU(2)$ group element which asymptotically coincides with U far from the z-axis going through the center of the loop perpendicular to its plane, and this regularized group element is to be non-singular

near this axis. Such a group element is given by the following expression

$$U = \exp(i\phi\tau_3/2)U_0\exp(i\phi\tau_3/2), \tag{5.12}$$

where

$$U_0(x,y,z) = \frac{1}{\sqrt{(r-x_0)^2\frac{r^2}{r^2+\alpha^2}+z^2}}\begin{pmatrix} (r-x_0)\frac{r}{r+i\alpha} & z \\ -z & (r-x_0)\frac{r}{r-i\alpha} \end{pmatrix} \tag{5.13}$$

Here α is a small parameter. This group element is odd under ordinary parity but is singular at the point $r = z = 0$. The singularity can be suppressed by assuming an appropriate zero of the profile function v at this point. In turn such a zero of the profile function does not correspond to any string linked with the original loop and does not spoil the pure gauge behaviour of the gauge field at infinity.

Let us now consider the U matrix given in Eq. (5.6) for the trivial loop configuration with $n = 0$, $m = -1$. The regularized version reads

$$U = \exp(i\phi\tau_3/2)\tilde{U}_0\exp(-i\phi\tau_3/2), \tag{5.14}$$

where

$$\tilde{U}_0 = \frac{1}{\sqrt{(r-x_0)^2+z^2\frac{r^2}{r^2+\alpha^2}}}\begin{pmatrix} r-x_0 & z\frac{r}{r+i\alpha} \\ -z\frac{r}{r-i\alpha} & r-x_0 \end{pmatrix}. \tag{5.15}$$

A similar modification can be used for general values of m and n. Regularized versions of Eqs. (5.6) and (5.8) can be obtained by substitution of the above regularized group elements and by the following substitution

$$e^{i\phi\tau_3} \rightarrow e^{i\phi\tau_3/2}(U_0)_{reg}e^{i\phi\tau_3/2} \cdot e^{-i\phi\tau_3/2}(U_0^{-1})_{reg}e^{i\phi\tau_3/2}, \tag{5.16}$$

where $(U_0)_{reg} = U_0$ and $(U_0^{-1})_{reg}(x,y,z) = \tilde{U}_0(x,y,-z)$. Here each factor is regular as was seen above. The resulting group element has a singularity at $r = z = 0$ which can be suppressed by introducing an appropriate zero of the profile function. Because it is odd under ordinary parity, the regularized versions of Eqs. (5.6) and (5.8), obtained by the above substitutions, have the same parity properties as the unregularized ones.

Let us now consider an electro-weak Z-string segment with a monopole and an anti-monopole ('t Hooft-Polyakov) attached to its ends [18]. The Z-string [17] is an embedded Nielsen-Olesen vortex, the core of which is a flux tube leading the electroweak Z field from the monopole to the anti-monopole. In our notation the gauge field for this configuration has the form (5.1) where v is an appropriate profile function and

$$U(x) = \begin{pmatrix} \cos(\Theta/2) & \sin(\Theta/2)e^{-i\phi} \\ \sin(\Theta/2)e^{i\phi} & -\cos(\Theta/2) \end{pmatrix}. \tag{5.17}$$

Here the angle variable $\Theta \in [0, \pi]$ is defined as follows

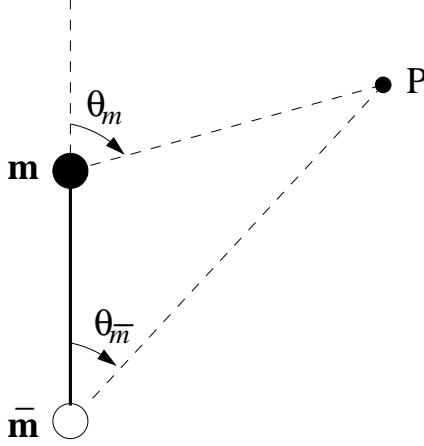$$\cos\Theta = \cos\theta_m - \cos\theta_{\bar{m}} + 1. \tag{5.18}$$

Figure 3: *Definition of polar angles in the Nambu monopole-antimonopole configuration.*

The polar angles θ_m and $\theta_{\bar{m}}$ are defined in Fig. 3. It can be seen that Θ is approximately the angle subtended by the string segment between the two monopoles at the position x. The U matrix in Eq. (5.17) corresponds to the Higgs field configuration

$$\Phi_{m\bar{m}} = f(x)\, U \begin{pmatrix} 1 \\ 0 \end{pmatrix} = f(x) \begin{pmatrix} \cos(\Theta/2) \\ \sin(\Theta/2)e^{i\phi} \end{pmatrix}, \qquad (5.19)$$

where $f(x)$ is a profile function.

One can check that the configuration given by Eqs. (5.17) and (5.18), in the limit $\theta_{\bar{m}} \to 0$, approaches the monopole configuration

$$U_{mon} = U_1(\theta_m, \phi) \equiv \begin{pmatrix} \cos(\theta_m/2) & \sin(\theta_m/2)e^{-i\phi} \\ \sin(\theta_m/2)e^{i\phi} & -\cos(\theta_m/2) \end{pmatrix}, \qquad (5.20)$$

and, similarly, that it approaches the anti-monopole configuration given by

$$U_{antimon} = U_1(\frac{\pi}{2} - \theta_{\bar{m}}, \phi) = \begin{pmatrix} \sin(\theta_{\bar{m}}/2) & \cos(\theta_{\bar{m}}/2)e^{-i\phi} \\ \cos(\theta_{\bar{m}}/2)e^{i\phi} & -\sin(\theta_{\bar{m}}/2) \end{pmatrix}. \qquad (5.21)$$

in the limit $\theta_m \to \pi$. The string singularity is localized on the straight line joining the monopole and anti-monopole and corresponds to $\theta_{\bar{m}} = 0$ and $\theta_m = \pi$. The angle ϕ is measured around this axis.

These U matrices correspond to the following Higgs fields near the monopole

$$\Phi_{mon} = f(x)U_1(\theta_m, \phi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = f(x) \begin{pmatrix} \cos(\theta_m/2) \\ \sin(\theta_m/2)e^{i\phi} \end{pmatrix}, \qquad (5.22)$$

21

and near the anti-monopole

$$\Phi_{antimon} = f(x)U_1(\frac{\pi}{2} - \theta_{\bar{m}}, \phi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = f(x)\begin{pmatrix} \sin(\theta_{\bar{m}}/2) \\ \cos(\theta_{\bar{m}}/2)e^{i\phi} \end{pmatrix}. \tag{5.23}$$

The parity transformation corresponds to

$$\phi \to \phi + \pi, \quad \theta_m \to \pi - \theta_{\bar{m}}, \quad \theta_{\bar{m}} \to \pi - \theta_m. \tag{5.24}$$

It is easy to see that the matrix U defined by Eq. (5.17) satisfies

$$U(-x) = -i\tau_3 U(x)i\tau_3, \tag{5.25}$$

and hence belongs to the trivial class 2 of our classification. Thus this configuration has Chern-Simons number 0 and does not permit fermionic level crossing. Note that this conclusion agrees with the analysis of Ref. [26].

Let us consider instead a "twisted" configuration, introduced in Ref. [27], of a monopole and an anti-monopole joined by a segment of electroweak string. The gauge field is given by the same formula (5.1) with

$$U_\gamma(\theta_m, \theta_{\bar{m}}, \phi) = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}, \tag{5.26}$$

where

$$a = \sin(\theta_m/2)\sin(\theta_{\bar{m}}/2)e^{i\gamma} + \cos(\theta_m/2)\cos(\theta_{\bar{m}}/2), \tag{5.27}$$
$$b = \sin(\theta_m/2)\cos(\theta_{\bar{m}}/2)e^{i\phi} - \cos(\theta_m/2)\sin(\theta_{\bar{m}}/2)e^{i(\phi-\gamma)}. \tag{5.28}$$

Here the constant γ is the angle of twist. This matrix U_γ corresponds to the following Higgs field

$$\Phi_{m\bar{m}} = f(x)\begin{pmatrix} a \\ b \end{pmatrix}. \tag{5.29}$$

It is easy to see that as $\theta_{\bar{m}} \to 0$ we get the monopole configuration with the U matrix given by Eq.(5.20) and as $\theta_m \to \pi$ one has the anti-monopole field with U defined in Eq.(5.21), provided we perform the rotation $\phi \to \phi + \gamma$. The configuration of Eq.(5.26) also has the usual string singularity along the line that joins the monopoles. Indeed let us take $\theta_{\bar{m}} = 0$ and $\theta_m = \pi$. In the domain near this line it is easy to see that the Higgs field reduces to

$$\Phi_{m\bar{m}} = \begin{pmatrix} 0 \\ e^{i\phi} \end{pmatrix} f(x). \tag{5.30}$$

Let us now consider the parity properties of the configuration given by Eqs. (5.1) and (5.26). It is easy to check that this configuration has a definite generalized parity only if $\gamma = 0$ or π. At $\gamma = 0$ the matrix U_γ is even under the ordinary parity transformation (5.24). Hence it corresponds to a topologically trivial field configuration connected to a vacuum; it has Chern-Simons number 0 and does not

permit fermionic level crossing. In contrast, for $\gamma = \pi$ the group-valued function U_γ is odd under parity. The corresponding configuration is topologically non-trivial and continuously related to a sphaleron field. It can be seen easily that, when the string joining the monopole and anti-monopole is collapsed to a point, the angles θ_m and $\theta_{\bar{m}}$ can be identified and we obtain the sphaleron configuration given in Eq. (2.2), up to a factor τ_3. Hence the configuration has Chern-Simons number $1/2$ and is relevant for rapid fermion number violation in the hot theory.

When we continuously change the angle of twist γ from 0 to π the configuration goes through fields which do not have any definite (generalized) parity eigenvalue. Therefore we cannot control the Chern-Simons number of the field under such a deformation. However we see that the Chern-Simons number changes continuously from 0 to $1/2$.

Let us now consider deformations, with definite generalized parity, of the above monopole-antimonopole configurations (with $\gamma = 0, \pi$). Such deformations can be constructed easily if we observe that for the trivial configuration ($\gamma = 0$)

$$U_{\gamma=0}(\theta_m, \theta_{\bar{m}}, \phi) = U_1(\theta_m, \phi)U_1(\theta_{\bar{m}}, \phi), \tag{5.31}$$

while for the non-trivial one ($\gamma = \pi$)

$$U_{\gamma=\pi}(\theta_m, \theta_{\bar{m}}, \phi) = U_1(\theta_m, \phi)\sigma_3 U_1(\theta_{\bar{m}}, \phi)\sigma_3, \tag{5.32}$$

where U_1 is defined in Eq. (5.20). Following the approach of Ref. [13], we shall see below that the monopole-antimonopole configurations may be continuously deformed into two segments of electroweak string, each segment joining a monopole with an anti-monopole.

Indeed let us consider the topologically trivial configuration with the $U_{\gamma=0}$ matrix. Define $\theta_{\bar{m}'}$ and $\theta_{m'}$ to be the polar angles measured from two points $\bar{m}'$ and m' located on the line between the initial monopole and anti-monopole (see Fig. 4). When both $\bar{m}'$ and m' coincide with the midpoint of this line, so that $\theta_{\bar{m}'} = \theta_{m'}$, we have

$$U_{\gamma=0}(\theta_m, \theta_{\bar{m}}, \phi) = U_{\gamma=0}(\theta_m, \theta_{\bar{m}'}, \phi) \times U_{\gamma=0}(\theta_{m'}, \theta_{\bar{m}}, \phi), \tag{5.33}$$

This follows from Eq. (5.31) and the identity $U_1^2 = 1$. Let us now deform the configuration continuously by moving the points $\bar{m}'$ and m' an equal distance in opposite directions along the axis of the string, so that $\theta_{\bar{m}'}$ and $\theta_{m'}$ are no longer equal. The points $\bar{m}'$ and m' are associated with an anti-monopole and a monopole which emerge at the midpoint and then separate. As a result the original string segment is divided into two segments; one between m and $\bar{m}'$, the other between m' and $\bar{m}$ (see Fig. 4). Such a splitting is easily arranged by a deformation of the profile function. In Eq. (5.33) the multiplication sign $\times$ (which is ordinary multiplication) separates the factors corresponding to the field configurations of the individual segments. The new configuration is even under a generalized parity transformation with respect to the midpoint between the two string segments. We have thus split the trivial configuration into two also trivial ones.
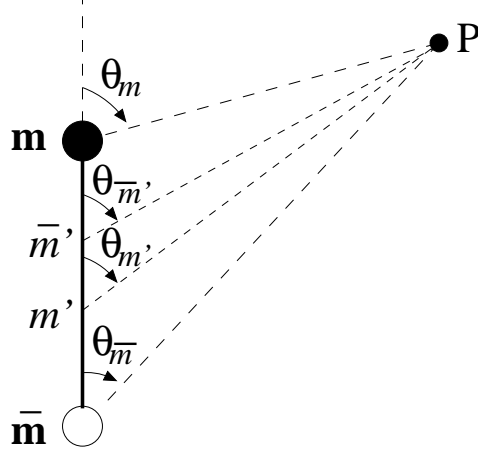
Figure 4: *The splitting of a Z-string segment, joining a monopole-antimonopole pair, into two Z-string segments by the production of a second monopole-antimonopole pair.*

One can also split the trivial configuration into two non-trivial ones as follows

$$
\begin{aligned}
U_{\gamma=0} &= U_1(\theta_m, \phi)\sigma_3 U_1(\theta_{\bar{m}'}, \phi)\sigma_3 \times \sigma_3 \left(U_1(\theta_{m'}, \phi)\sigma_3 U_1(\theta_{\bar{m}}, \phi)\sigma_3 \right)\sigma_3 \\
&= U_{\gamma=\pi}(\theta_m, \theta_{\bar{m}'}, \phi) \times \sigma_3 U_{\gamma=\pi}(\theta_{m'}, \theta_{\bar{m}}, \phi)\sigma_3.
\end{aligned}
\tag{5.34}
$$

Again the equality holds when $\theta_{\bar{m}'} = \theta_{m'}$ by virtue of Eqs. (5.31), (5.32) and $U_1^2 = 1$. As before, we split a single string segment into two by moving $\bar{m}'$ and m' in opposite directions from the midpoint of the string and deforming the profile function accordingly. The new configuration is obviously even under a generalized parity transformation with respect to the midpoint between the two segments and is topologically trivial, while each segment corresponds to a topologically non-trivial configuration.

This particular example shows clearly that there exist configurations which globally admit only an even number of zero modes of the Dirac equation, but consist of localized excitations each corresponding to a saddle-point-like configuration with an odd number of zero modes which can mediate $B + L$ violating transitions. This illustrates the concept of a fluid composed of localized $B + L$ violating excitations that was alluded to in the introduction. The properties of such a "fluid" require further investigation. We shall return to this issue in the next section.

Similarly we can split the nontrivial configuration given by $U_{\gamma=\pi}$ into one non-trivial and one trivial configuration along the axis of the string as follows.

$$
U_1(\theta_m, \phi)\sigma_3 U_1(\theta_{\bar{m}}, \phi)\sigma_3 = U_1(\theta_m, \phi)U_1(\theta_{\bar{m}'}, \phi) \times U_1(\theta_{m'}, \phi)\sigma_3 U_1(\theta_{\bar{m}}, \phi)\sigma_3.
\tag{5.35}
$$

We separate $\bar{m}'$ and m' along the string axis and get a new configuration, odd under a generalized parity transformation with respect to the midpoint between the two string segments.

We have thus described a set of deformations, with definite parity, of the monopole-antimonopole configuration. This analysis, together with that of Ref. [13] for the loop configurations, provides a list of parity-preserving deformations of the string loops and string segments at the classical level. This list includes an infinite set of odd-parity deformations of the sphaleron which, like the sphaleron, mediate baryon-number changing processes near and above the electroweak phase transition temperature. The analysis of the processes of splitting can be instructive for understanding the dynamics of string loops and segments at the quantum level.

We conclude this section by observing that the *new* electroweak "sphaleron" S^* fits into our classification of gauge fields. Its gauge-field configuration is defined by Eq. (5.1) where the matrix U, extracted from Ref. [8] with some amount of algebra, is given by

$$U = i \, \frac{2zx_i\tau_i - \left[r^2 + z^2 + \left(\frac{d}{2}\right)^2\right]\tau_3}{\sqrt{\left(r^2 + \left(z - \frac{d}{2}\right)^2\right)\left(r^2 + \left(z + \frac{d}{2}\right)^2\right)}}. \tag{5.36}$$

The profile function $v = v(r, z)$ satisfies $v(r, z) = v(r, -z)$ and has two zeros at the positions $r = 0$, $z = \pm d/2$. These zeros can be thought of as the loci of a sphaleron S and an anti-sphaleron $\bar{S}$ which together form the bound state S^*. The gauge fields constructed from U and Eq. (5.1) are obviously odd under ordinary parity. Because U is even, the configuration S^* is topologically trivial and plays no role in $B + L$ violating processes.

Similar analysis can be done for the multisphaleron configurations with $CS = n/2$, $n \in \mathbf{Z}$. For the axisymmetric ansatz given in [9], multisphalerons with n odd have odd U at infinity, possess an odd number of fermion zero modes and thus mediate $B + L$ violation. Because each class of the $\mathbf{Z}_2$ homotopy group is non-empty, we expect the existence of even-n ($CS \in \mathbf{Z}$) multisphaleron solutions with (generalized)-odd gauge fields, even U at infinity, possessing an even number of fermion zero modes. Such multisphalerons are connected to the S^* new sphaleron and the trivial vacua and do not mediate $B + L$ violation in the early universe.

6   Conclusions and Discussion

In the present work we have addressed the issue of the topological origin of $B+L$ violation in the electroweak theory. The existence of a homotopy group $[\mathbf{R}P^2, G/\mathbf{Z}_2] = \mathbf{Z}_2$ for $G = SU(2N)$, $SO(2N)$ and E_7 implies the existence of a $\mathbf{Z}_2$ structure in the gauge sector of the electroweak theory in its symmetric phase as well as its broken phase. Because of this $\mathbf{Z}_2$ structure the Chern-Simons number (CS) plays the new role of a topological index for gauge fields which are odd under a generalized parity transformation. For configurations with even-parity pure-gauge behaviour at infinity, CS takes an integer value n, while for an odd-parity pure-gauge behaviour, CS takes a value $n + 1/2$. Thus group theory implies the existence of two topologically distinct infinite sets of odd-parity gauge field configurations with a continuous range

of energies. This structure characterizes both the symmetric and the broken phase of the theory and does not depend on the particular Higgs sector of the model. In the broken phase, it automatically accounts for the existence of a sphaleron saddle point with energy $E_{sph} \approx M_W/\alpha$ and Chern-Simons number $CS = 1/2$.

The important role of sphalerons and their deformations in $B + L$ violating processes is a direct consequence of the topological invariance modulo 2 of the number of zero modes of the Dirac operator in their background. This invariance implies that $B + L$ violating fermion level crossings occur in the presence of *any* finite-energy (generalized) odd-parity gauge field configuration with an odd pure-gauge behaviour at infinity ($CS = n + 1/2$). Odd-parity gauge field configurations with an even pure-gauge behaviour at infinity, such as the new S^* "sphaleron", admit an even number of fermion zero modes and do not mediate $B + L$ violation.

We remark that, while the $\mathbf{Z}_2$ topological structure provides us with an infinity of gauge-Higgs field configurations with $B+L$ violating properties, we cannot claim to have exhausted the whole space of such configurations. In fact, consider a path leading from the vacuum with $CS = n$ to the vacuum $CS = n + 1$ over some odd-parity configuration with $CS = n + 1/2$. It is easy to show from continuity of the Dirac spectrum that if a small even-parity gauge field A_E is added to every configuration along the path, satisfying $A_E(n) = A_E(n+1) = 0$, then an odd number of zero modes must occur for some Chern-Simons number $n < CS < n + 1$.

Despite these findings, we assert that the space of generalized odd-parity gauge fields with odd pure-gauge behavior at infinity constitutes the backbone in the body of all configurations that mediate $B + L$ violation. This is quite transparent in the broken phase of the theory ($T < E_{sph}$), where the sphaleron saddle point *alone* governs the dynamics of thermal transitions between vacua.

As the temperature is raised, other configurations will begin to contribute to the thermal transition rate. Those with odd generalized parity are the first ones to become excited. This result comes about from studying configurations in a neighborhood of the sphaleron. More precisely, we consider the space E of deformations of the sphaleron which are orthogonal to the unstable mode. Let us introduce an $\mathcal{L}^2$ metric on E and consider a sphere $S_\epsilon \in E$ with radius ϵ centered at the origin (which corresponds to the sphaleron). On this compact sphere, the energy functional assumes maximal and minimal values. Because the energy functional is invariant under the generalized parity transformation defined in section 3.1, fields A_i which are invariant under this discrete symmetry correspond to an extremum on S_ϵ. It is natural to expect, by continuity arguments, that the extremum constitutes a minimum. This claim can be justified through direct computation.

As the result of such an argument, on the sphere S_ϵ sphaleron deformations with odd generalized parity have lower energy than the rest. Therefore, as the temperature is raised, these sphaleron deformations are the first to be excited. The concept that generalized odd-parity configurations with odd pure-gauge behavior at infinity are the backbone of the configurations mediating $B + L$ violation is thereby justified, at least for temperatures near and not too high above the sphaleron energy

E_{sph}.

It is appropriate at this point to remark about the domain of validity of our arguments. In an $SU(2)$ gauge theory it was shown [10] that for $M_H > 10 M_W$ there emerges a doublet of saddle-point configurations of lower energy than the $CS = 1/2$ sphaleron with $CS = 1/2 \pm \epsilon(M_H)$. For such a range of Higgs mass, and at temperatures comparable to the energy of the saddle points, we would expect the $B + L$ violating transitions to be dominated instead by gauge-field configurations with $1/2 - \epsilon < CS < 1/2$ and $1/2 < CS < 1/2 + \epsilon$.

While the density of $B + L$ violating sphaleron-like configurations is Boltzmann suppressed for $T < E_{sph}$, the expected monotonic increase of density of such configurations as a function of temperature would provide an explanation for the unsuppressed $B + L$ violating transition rate in the symmetric phase ($\Gamma \propto T^4$), which cannot be understood from perturbative expansions around the sphaleron [4, 28].

We may further speculate that the large density of finite-energy configurations homotopically equivalent to the sphaleron will induce dynamically a $B + L$ violating quark-lepton condensate in the symmetric phase. The analogous phenomenon, in the presence of a nonzero density $\rho(0)$ of instanton energy levels near zero energy, has already been studied in the context of chiral symmetry breaking in QCD [29], where the quark condensate $\langle \overline{\psi} \psi \rangle$ is proportional to $\rho(0)$. In our case the magnitude of the condensate would be proportional to the density of energy levels of the three-dimensional Dirac operator near zero energy. At high temperature the condensate has to be proportional to a power of the temperature since this is the only dimensionful parameter.

The form of such a fermionic condensate is restricted by the gauge symmetry $SU(3)_c \times [SU(2) \times U(1)]_{EW}$ of the standard model. The simplest form is

$$\langle \sum_{ijk} \epsilon_{ijk} (q_L^i q_L^j)(q_L^k l_L) \rangle, \tag{6.1}$$

where q_L and l_L are the quark and lepton left-handed doublets and the indices i, j, k stand for the fundamental representation of the $SU(3)_c$ group. It is easy to see that the above expression has zero hypercharge and hence is gauge invariant. Since we have no direct experimental information on whether color symmetry could have been spontaneously broken at high temperature, we should not exclude color-charged condensates from consideration. The possible existence of $B + L$ violating condensates in the symmetric phase will be the subject of a future investigation.

A brief summary of the results of this paper was given in Ref. [30].

7 Acknowledgments

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